PHYSICAL REVIEW
LETTERS

VOLUME 79 8 DECEMBER 1997 NUMBER 23

Stern-Gerlach Effect for Electron Beams

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(Received 27 May 1997)

The conflict between Bohr's assertion that the magnetic moment of the electron cannot be measured with experiments based on the concept of classical trajectories, and the measurement of the magnetic moment of electrons in a modified Penning trap by Dehmelt *et al.* has led us to reevaluate other implications of Bohr's assertion. We show that, contrary to the analysis of Bohr and Pauli, the assumption of classical trajectories in a Stern-Gerlach–like device can result in a high degree of spin separation for an electron beam. This effect may persist within a fully quantum-mechanical analysis. The magnetic fields considered are such that a tabletop Stern-Gerlach electron spin filter is feasible. [S0031-9007(97)04768-6]

PACS numbers: 03.65.Bz, 14.60.Cd, 34.80.Nz

In the early years of quantum mechanics, several of its inventors debated at length if a Stern-Gerlach magnet could be used to polarize electrons beams. Stern and Gerlach, in their famous experiment, had demonstrated such a device for atoms. The problem with using a standard Stern-Gerlach magnet for electrons is that the splitting is completely blurred by the Lorentz force acting on a beam of finite transverse dimensions [1]. Brillouin suggested an alternate experiment in which the electrons were separated by spin using magnetic gradient forces acting along the direction of motion, instead of transversely to it [2]. This approach, however, was declared unsound by Bohr's assertion, as formulated by Pauli, that "*it is impossible to observe the spin of the electron, separated fully from its orbital momentum, by means of experiments based on the concept of classical particle trajectories*" [3]. At the sixth Solvay conference Pauli, supported by Bohr, explicitly rejected Brillouin's proposal as well as three others [4]. Any attempt to turn a thought experiment into a real one was thus discouraged at an early stage. The Bohr/Pauli arguments have been codified in many textbooks and monographs, and today it is widely accepted that it is impossible to construct an electron-polarizing beam splitter that uses macroscopic electromagnetic fields [5–8].

Recently, however, the range of validity of Bohr's assertion has been rendered uncertain by the beautiful experiments of Dehmelt and his colleagues, in which electrons of a given spin are isolated in a modified Penning trap, allowing a determination of their magnetic moment μ_B [9]. In view of this, we feel it important to study further the range of validity of Bohr's edict. The explicit use of the phrase "classical trajectories" in his arguments makes useful any counter-argument based on such concepts, none of which exist to our knowledge.

Thus in this Letter, we address the specific question of whether one can, while considering classical particle trajectories, separate an electron beam by spin, using an apparatus based on magnetic fields alone. Such systems are conceptually simple and analogous to the traditional Stern-Gerlach situation and Brillouin's thought experiment. Moreover, since they involve beams of electrons, they could be used as sources or analyzers of polarized electrons. An example similar to Brillouin's original proposal is considered and yields a complete separation of the electron spins. This result contradicts Pauli's rejection of Brillouin's proposal, which is based on an oversimplified approximation of the electron trajectories. A full quantum-mechanical treatment of this problem is beyond the scope of this Letter, but we present a brief argument that a spin splitting of the electrons, albeit reduced, will persist in an electron wave treatment. A device based on our analysis is experimentally feasible.

In Brillouin's proposal [2], electrons with a precise energy are injected into an inhomogeneous magnetic field

at some angle to the primary direction of the field, which we choose to be along the z axis. The kinetic energy of the electrons associated with their velocity along $\hat{z}$ depends on this angle, and the potential energy depends on their spin projection along the same direction. Electrons with spins parallel to the field require a different minimum insertion angle than those with spins antiparallel to it if they are to reach the magnet pole piece that generates the field and be detected. Determination of these minimum angles in effect determines the Bohr magneton. Alternately, this scheme can be viewed as separating electrons by spin.

We now paraphrase the Bohr/Pauli refutation of Brillouin's proposal [4]. Consider electrons moving parallel to the z axis and antiparallel to the primary magnetic field direction. If $\partial B_z/\partial z > 0$, then electrons with their spins parallel to the z axis with initial speed v_z stop and reverse direction within a time t given by $mv_z = \mu_B(\partial B_z/\partial z)t$, where m is the electron mass. The number of electrons that cover a distance along z greater than $v_z t$ is half the number that would cover this distance in the absence of spin. Now suppose that the magnetic field is parallel everywhere to the xz plane, so that $\partial B_x/\partial x = -\partial B_z/\partial z$. If the field at $x = 0$ is exactly along $\hat{z}$, then at a distance Δx from the z axis the magnetic field component B_x is given by $B_x = (\partial B_x/\partial x)\Delta x = -(\partial B_z/\partial z)\Delta x$. This field causes the velocity in the z direction to *reverse sign* (our italics) in the Larmor precession time. If the device is to separate trajectories based on field-gradient spin forces, then such forces must act over a time much less than the Larmor precession time, i.e., $t \ll h/\mu_B B_x$ must hold or, equivalently, $\mu_B(\partial B_z/\partial z)t\Delta x \ll h$, which reduces to $mv_z\Delta x \ll h$.

Because of the wave nature of the electron this last condition (though formally different from the uncertainty relation) cannot be satisfied during the complete interaction time, because the de Broglie wavelength λ is just h/mv_z, and beam widths Δx such that $\Delta x \ll \lambda$ are not possible. Should one attempt to create such a beam with an aperture, then $\Delta v_x > h/(m\Delta x)$ by the uncertainty principle, which requires $\Delta v_x \gg v_z$, and the outcome of the experiment cannot be predicted by classical mechanics. Here Pauli's central argument stops.

This reasoning is questionable for it incorporates an illegitimate approximation of the actual classical trajectories. Although an electron slightly displaced from the z axis will experience a force that starts to rotate its velocity towards the y axis, the change in the direction of motion is modified by the small induced y component of velocity which in turn causes a Lorentz force due to the magnetic field along $\hat{z}$. The resulting trajectory is the familiar helical spiral around the direction of the magnetic field, with only one direction of motion resulting along the z axis.

Dehmelt has disputed Pauli's more general 1932 "proof" of Bohr's assertion [9,10]. Pauli showed that an expansion in terms of $\hbar$ for solutions of the Dirac equation leads to the conclusion that "*all effects on the density and flow of the particles due to the spin appear automatically in the same order of approximation as effects due to diffraction of the matter waves*." This means that both the spatial split up of the two spin states of an electron beam by a Stern-Gerlach magnet, and the blurring of this split up due to diffraction, are proportional to $\hbar$. Bohr's assertion is thus based on "taking the classical limit" $\hbar \to 0$. For this limit not only the blurring, but also the Stern-Gerlach splitting vanishes. However, Dehmelt argues that $\hbar$ is a nonzero constant of nature, and that the classical limit must be approached in other ways. He proposes three criteria for achieving this: (a) using an apparatus in which the Lorentz forces are minimized, (b) maximizing the spin forces by using large magnetic field gradients, and (c) eliminating wave effects by using an apparatus whose characteristic dimensions are much larger than the electron wave packet. The example given by Dehmelt of an experiment satisfying these conditions involves a combination of an electric and magnetic field, and is analyzed in terms of quantum-mechanical states. Moreover, he does not address the completeness of the separation of the spin components [11,12], making a comparison with Bohr's assertion even less direct, and the implications for a Stern-Gerlach–like beam apparatus less obvious.

Still, Dehmelt's three criteria seem generally applicable, and we study their usefulness in an example with an electron beam in a pure magnetic field, using classical particle trajectories. The essential problem with the transverse Stern-Gerlach geometry is that large Lorentz forces act on a charged-particle beam. To eliminate this problem we propose, like Brillouin, that longitudinal fields be used. The application of ancillary electric fields appears to be unnecessary to realize a successful spin filter [4]. We consider two geometries: that of a two-wire field [Fig. 1(a)] in which the plane containing the wires is perpendicular to the primary electron beam axis, and that of a solenoid [Fig. 1(b)], whose axis of symmetry is along the beam direction.

We now describe our semiclassical approach to the behavior of the electrons in a magnetic field. The electron spin is treated quantum mechanically as required by the nature of the problem. As usual, the amplitudes a_i for the spinors a are given in terms of the magnetic field

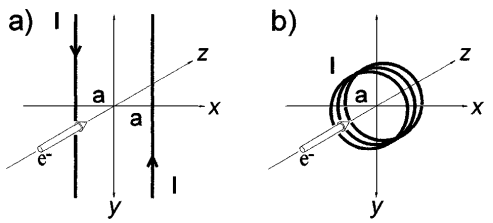

FIG. 1. (a) Schematic drawing of the two-wire longitudinal "Stern-Gerlach" geometry. Beam splitting occurs along the z axis. (b) The solenoid geometry; splitting is also along $\hat{z}$.

components by

$$i\hbar \frac{d}{dt}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \mu_B \begin{pmatrix} B_z & B_x + iB_y \\ B_x - iB_y & -B_z \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \quad (1)$$

The energy eigenvalues that are used below in the treatment of the motion are $E_\pm = \pm\mu_B\sqrt{B_x^2 + B_y^2 + B_z^2}$. In all simulations discussed here, the spin-flip probability was found to be less than 10^{-3} per electron. The strong adiabaticity of the electron spin precession in the slowly changing magnetic fields we considered effectively prevents spin flips from occurring.

The electron motion is treated semiclassically. Comparing the de Broglie wavelength of the electrons ($\sim$1 nm) with the typical size ($\sim$1 cm) of the proposed geometries seems to justify this approach. Moreover, in view of the Bohr phrase "particle trajectories" and Pauli's use of classical trajectories in his rejection of Brillouin's proposal, such a treatment is appropriate.

Following Pauli's approach, the spread in the initial conditions for the electron trajectories was chosen to be consistent with the Heisenberg uncertainty relations. We used initially Gaussian spatial and velocity distributions, with respective widths $\Delta\vec{r}_i$ and $\Delta\vec{v}_i$, with an average velocity $\vec{v}_z$. The choice $\Delta\vec{r}_i = \Delta\vec{v}_i t$, where t is the electron flight time, minimizes the geometric beam spread in the absence of fields. In combination with the uncertainty principle, it determines the set of initial conditions of the trajectories.

The force acting on the electrons due to the magnetic field is given by the sum of the Lorentz force and the spin force $\vec{F}_{s,\pm} = -\vec{\nabla}E_\pm$, which is connected to the quantum-mechanical description of the spin by the use of the energy eigenvalues $E_\pm$. For particles confined to the z axis, we can calculate analytically the spatial separation of the two spin components Δz_{spin} from

$$\begin{aligned}\Delta z_{\text{spin}} &= \int\int \frac{1}{m}\frac{\partial(E_+ - E_-)}{\partial z}\,dt'\,dt \\ &= \int\int \frac{2\mu_B}{m}\frac{\partial B_z}{\partial z}\,dt'\,dt\,.\end{aligned} \quad (2)$$

For general particle trajectories, we have numerically integrated the equations of motion. The trajectories are determined using either E_+ or E_-. Both the equation of motion and the equation for spin are integrated simultaneously to obtain the trajectory of the electron and its spin-flip probability.

Now we consider the first case of an electron beam along the z axis passing through the middle of two current-carrying wires [Fig. 1(a)]. The wires run parallel to the y axis a distance $\pm a$ from it and each carries a current I flowing in opposite directions. Apart from the fact that the resulting magnetic field is directed primarily along the electron beam direction, thus minimizing Lorentz forces, the choice of this geometry is based on two considerations. First, its two-dimensional nature recalls Pauli's argument above, and second, the analytic expression for the magnetic field over all space is simple. To illustrate our results, we take the initial positions of the electrons along the direction of motion to be one meter away from the wires, v_z to be 10^5 m/s, and $a = 1$ cm. The equations of motion were integrated for 20 μs (corresponding to a 2 m flight path) with a Gaussian distribution of starting positions of 80 μm FWHM along all three axes, and with a three-dimensional Gaussian distribution in the starting velocities of 4.0 m/s FWHM, satisfying Heisenberg's uncertainty relations [13] and corresponding to the spreading minimization criterion mentioned above. The field strength exactly between the wires, B_0, is taken to be 10 T. The spatial separation of the two spin components can be estimated by evaluating Eq. (2). The result for an on-axis trajectory from z_i to z_f is

$$\Delta z_{\text{spin}} = \frac{2a\mu_B B_0}{m v_z^2} \times \left[\tan^{-1}(z_f/a) - \tan^{-1}(z_i/a) - \frac{(z_i - z_f)a}{a^2 + z_i^2}\right], \quad (3)$$

which for the above parameters equals 631 μm, in agreement with our numerical simulation shown in Fig. 2(a).

The results of this calculation offer a direct counterexample to Pauli's arguments. Our simulation shows that the electrons behave in the magnetic field in a manner qualitatively similar to that predicted by Brillouin; they execute spiral trajectories about the pinched field lines with decreasing helical radii as they approach the wires. This illustrates the conceptual problem with Pauli's argument paraphrased above. While the on-axis spin splitting is

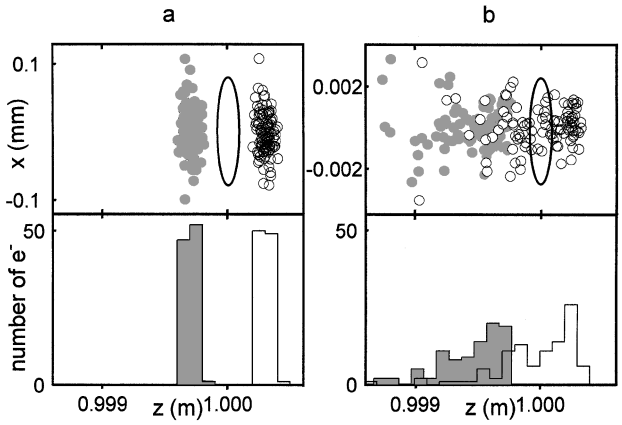

FIG. 2. (a) The spatial distribution of electrons before and after passage through the two-wire field. Open circles represent "spin-up" electrons; solid circles represent "spin-down" electrons. The solid line encloses approximately 90% of the electrons in their initial Gaussian spatial distribution (see text) inserted 1 m upstream of the wires. (b) Same as above but with initial conditions corresponding to the $n = 0$ Landau state. Note the scale change, and the fact that the Landau $n = 0, m_s = -1/2$ electrons are not accelerated longitudinally.

derivable analytically, the numerical simulation is crucial to show that no off-axis effects blur or reduce the splitting.

Can such spin splitting be expected to persist in a quantum-mechanical calculation? We present an argument in favor of an affirmative answer. In a homogeneous magnetic field along $\hat{z}$ the Landau energy eigenvalues for an electron are given by [14]

$$E_n = p_z^2/2m + (2n + 1)\mu_B B \pm \mu_B B; \quad n = 0, 1, 2, \ldots . \tag{4}$$

Both the second and third terms of Eq. (4) result in forces acting on the electrons when magnetic field gradients are present. The spin force we have used in the simulations above is the gradient of the last term. The gradient of the second term corresponds to a "magnetic bottle" force associated with the transverse motion of the electrons. Classically, as the field strength increases, the electrons spiral in increasingly tighter orbits, losing longitudinal kinetic energy in the process. In the simulation above, these forces were negligible compared with the spin force. Quantum mechanically, however, one is not free to choose any Δr_i and Δv_i that satisfy Heisenberg's uncertainty principle, as we did in the above simulation and as Pauli did. Instead, we must pick initial distributions that match those of Landau states. Even the $n = 0$ Landau level has a transverse momentum distribution much broader than that associated with the minimum spreading criterion used above, and the magnetic bottle forces are correspondingly larger. Running our simulation with initial conditions appropriate for the lowest Landau state yields a blurring of exactly the same size as the split up of the two electron spin components, as one would expect from the equal magnitude of the second and third terms in Eq. (4). These results are shown in Fig. 2(b). It is apparent that when one uses initial conditions dictated by quantum mechanics, the spin splitting by the magnetic field is blurred significantly. The important point here, though, is that the beam splitting is still clearly evident and not marginal. This is in marked contrast to the generally accepted interpretation of the Bohr/Pauli assertion and to the case of electron deflection by a transverse Stern-Gerlach magnet, where the spin blurring is essentially complete [15].

Though the two-wire geometry illustrates the problem with Pauli's argument, it would be impossible to realize experimentally; to obtain the spin separation of the beam shown in Fig. 2(b), we used $I = 10^5$ A. To reduce this current, we also considered the behavior of an electron beam traveling close to the symmetry axis of a solenoid with radius $a = 1$ cm. The spin separation [Eq. (2)] for electrons traveling exactly on the axis of symmetry ($r = 0$) over a distance much greater than the solenoid radius is

$$\Delta z_{\text{spin}} = \frac{2L\mu_B\mu_0 nI}{mv_z^2}, \tag{5}$$

where L is the solenoid length and n is the linear winding density. This means that the required current for a 0.5 m long solenoid with 10 000 turns and an inner diameter of 1 cm is 5 A. Chopping an electron beam at a frequency of 1 GHz would allow a separation of the spin components to be observed with a time resolution of 1 ns. The low-velocity tail of a 10 eV, 1 μA beam transversely collimated through two 1 μm apertures 2 mm apart would yield an estimated signal of 0.3 electrons per second. The results of our simulation for this geometry agree with Eq. (5) and are essentially identical to those shown in Fig. 2(b).

In summary we have presented a semiclassical analysis of an electron beam passing through an inhomogeneous magnetic field. The main results and conclusions of this work are the following: (a) The outgoing beam has complete spatial separation of the electron spin components, (b) the Bohr-Pauli analysis of Brillouin's thought experiment is incorrect, (c) Bohr's general assertion concerning observation of electron spin is not universally applicable, (d) a provisional estimate of the quantum-mechanical result shows that the spin splitting is blurred to the same order as the splitting itself, but that nonnegligible polarization effects are still extant, and (e) our geometries, chosen in accordance with Dehmelt's three criteria, indicate that it is reasonable to attempt the design of a Stern-Gerlach device for an electron beam.

This work was supported by NSF Grant No. PHY-9504350. J. J. Schwendiman acknowledges support by the NSF-REU program. The authors thank G. A. Gallup, P. Finkler, W. C. Moss, and P. S. Farago for many helpful discussions.

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